Cyclic refinement of inequality $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9.$

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Let a,b and c be positive real numbers. Prove that

$$27abc(a^2b+b^2c+c^2a) \leq (a+b+c)^2(ab+bc+ca)^2.$$

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Note that $27abc(a^2b+b^2c+c^2a) \leq (a+b+c)^2(ab+bc+ca)^2 \iff$

(**1**) $27\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right) \leq (a+b+c)^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$

and sinse $\frac{a}{c}+\frac{b}{a}+\frac{c}{b} \geq 3$ then (**1**) implies inequality $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$ and can be considered as it's refinement.

Also note that transposition a with b transform $\frac{a}{c}+\frac{b}{a}+\frac{c}{b}$ to $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ but not changed expressioin $(a+b+c)^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$.

Hence, inequality (**1**) holds for any positive a,b,c if it holds for any positive a,b,c such that $\frac{a}{c}+\frac{b}{a}+\frac{c}{b} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$.

Since $\frac{a}{c}+\frac{b}{a}+\frac{c}{b} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \iff (b-c)(a-b)(a-c) \geq 0$ for positive a,b,c then in assumption $a = \max\{a,b,c\}$ (due to cyclic symmetry of (1)) suffices prove inequality (**1**) for $a \geq b \geq c > 0$. In that case denoting $p := \frac{a}{b} \geq 1, q := \frac{b}{c} \geq 1$ we obtain $b = qc, a = pqc, p,q \geq 1$ and then inequality (**1**) becomes

(**2**) $27\left(pq+\frac{1}{p}+\frac{1}{q}\right) \leq (pq+q+1)^2\left(\frac{1}{pq}+\frac{1}{q}+1\right)^2, p,q \geq 1.$

Let $u := p+q \geq 2, v := pq \geq 2$. Then we obtain

$$(pq+q+1)^2\left(\frac{1}{pq}+\frac{1}{q}+1\right)^2 - 27\left(pq+\frac{1}{p}+\frac{1}{q}\right) =$$

$$\frac{(p+q)(2p^3q^3+8p^2q^2-19pq+2)+(pq+1)^2(p+q)^2+p^4q^4-21p^3q^3+11p^2q^2+6pq+1}{p^2q^2} =$$

$\frac{u(2v^3+8v^2-19v+2)+(v+1)^2u^2+v^4-21v^3+11v^2+6v+1}{p^2q^2} \geq 0$ (because

$2v^3+8v^2-19v+2 = (v-1)(2v^2+10v-9)-7 \geq -7, v^4-21v^3+11v^2+6v+1 \geq -2, (v+1)^2 \geq$

and, therefore, $u(2v^3+8v^2-19v+2)+(v+1)^2u^2+v^4-21v^3+11v^2+6v+1 \geq$

$4u^2-7u-2 = (4u+1)(u-2) \geq 0$).